Combining Data for Improved Modeling

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Outline

- Inverse Methods
- Near subsurface imaging of the Earth
- Electrical Resistivity Tomography using Tikhonov regularization
  - Regularization informed by structural constraints
- Assessing effectiveness of combining different data types
- Results from combining Electrical Resistivity and Ground Penetrating Radar data

Inverse Problems

$$\mathbf{Gm} = \mathbf{d}$$

- $\mathbf{G} \in R^{m \times n}$ - mathematical model
- $\mathbf{d} \in R^{m}$ - observed data
- $\mathbf{m} \in R^{n}$- unknown model parameters

Least squares estimates

$$\mathbf{m}_{\mathsf{LS}} = \arg\min_{\mathbf{m}} \ \|\mathbf{G}\mathbf{m} - \mathbf{d}\|_2^2$$

If $\mathbf{G}$ has linearly independent columns

$$\mathbf{m}_{\mathsf{LS}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$$

In many practical applications $\mathbf{G}$ is rank deficient.

Regularization

$$\mathbf{m}_{\mathbf{L}_p} = \arg\min_{\mathbf{m}} \left\{ ||\mathbf{G}\mathbf{m} - \mathbf{d}||_2^2 + \alpha^2 ||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \right\}$$

$\mathbf{m}_0$ - initial estimate of $\mathbf{m}$

$\mathbf{L}_p$ - typically represents the first $(p = 1)$ or second derivative $(p = 2)$

α - regularization parameter

This gives estimates

$$\mathbf{m}_{\mathbf{L}_p} = \mathbf{m}_0 + (\mathbf{G}^T\mathbf{G} + \alpha^2\mathbf{L}_p^T\mathbf{L}_p)^{-1}\mathbf{G}^T\mathbf{d}$$

Choice of regularization parameter

Methods: L-curve, Generalized Cross Validation (GCV) and Morozov's Discrepancy Principle, UPRE, χ^2 method[1].

- α large $\rightarrow$ $\arg\min_{\mathbf{m}} \|\mathbf{L}_p(\mathbf{m}-\mathbf{m}_0)\|_2^2$

  $\mathbf{L}_p\mathbf{m} \approx \mathbf{0}$, i.e. $\mathbf{m}_{\mathbf{L}_p}$ is smooth

- α small $\rightarrow$ $(\mathbf{G}^T\mathbf{G} + \alpha^2\mathbf{L}_p^T\mathbf{L}_p)^{-1}$ DNE

  problem may stay ill-conditioned

[1]Mead et al, 2008, 2009, 2010, 2016

Choice of L_p

$$\mathbf{m}_{\mathbf{L}_p} = \arg\min_{\mathbf{m}} \left\{ \|\mathbf{G}\mathbf{m} - \mathbf{d}\|_2^2 + \alpha^2 ||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \right\}$$

$\mathbf{L}_0 (= \mathbf{I})$ - requires good initial estimate $\mathbf{m}_0$, otherwise may not regularize the problem.

$\mathbf{L}_1$ - requires first derivative estimate $\mathbf{L}_1\mathbf{m}_0$, i.e. changes in $\mathbf{m}_0$, which is less information than $\mathbf{m}_0$.

$\mathbf{L}_2$ - requires $\mathbf{L}_2\mathbf{m}_0$, leaves more degrees of freedom than first derivative so that data has more opportunities to inform changes in parameter estimates.

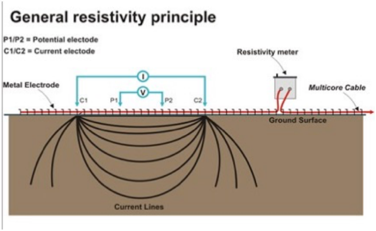
Near subsurface imaging

Boise Hydrogeophysical Research Site (BHRS)

- Field laboratory on a gravel bar adjacent to the Boise River, 15 km southeast of downtown Boise.
- Consists of coarse cobble and sand. Braided stream fluvial deposits overlie a clay layer at about 20 m depth.

Difference in retention properties in a lenticular sand feature yields significantly different geophysical properties.

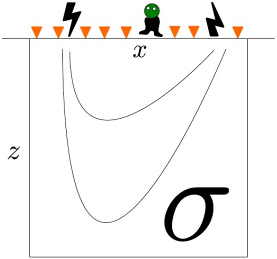
Electrical Resistivity Tomography (ERT)



- 2D grid of observations provides a 2.5-D inverted model that emphasizes the sand lenticular feature.
- BHRS survey consisted of 12 electrodes spaced 1 meter apart acquired with a dipole-dipole configuration.

BHRS survey acquired at surface when subsurface achieved saturation.

Electrical Resistivity Model



$$-\nabla \cdot \sigma \nabla \varphi = \mathbf{i}(\delta(x - s_+) - \delta(x - s_-))^2$$

φ - electric potential

$\mathbf{i}$ - current intensity

$s_\pm$ - source-sink position.

Model parameters: conductivity σ or resistivity $\rho = 1/\sigma$

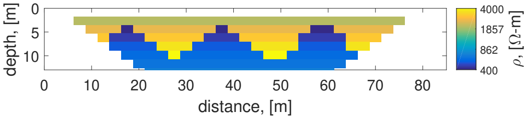
Observed data: apparent resitivity $\frac{2\pi\triangle\varphi}{i}\kappa$

[1] Pidlisecky and Knight, 2008

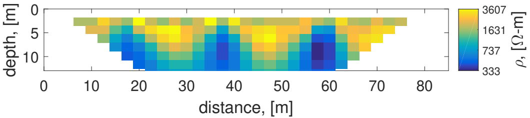
Simulated ERT results

$$\min_{\boldsymbol{\sigma}} \left\{ \|\mathbf{d} - \mathbf{G}(\boldsymbol{\sigma})\|_2^2 + \alpha^2 \|\mathbf{L}_p \boldsymbol{\sigma}\|_2^2 \right\}$$

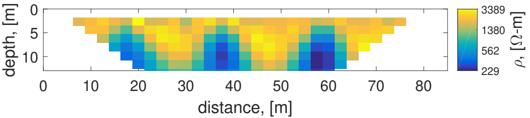
True resistivity



Inverted resistivity with L_1



Inverted resistivity with L_2

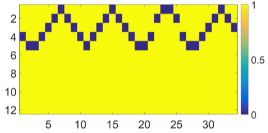


Structural Constraint

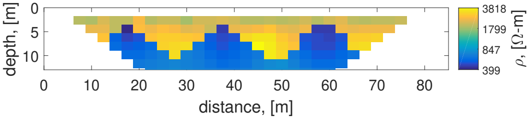
$$\min_{\boldsymbol{\sigma}} \left\{ \|\mathbf{d} - \mathbf{G}(\boldsymbol{\sigma})\|_2^2 + \alpha^2 \|\mathbf{R}\mathbf{L}_p \boldsymbol{\sigma}\|_2^2 \right\}$$

$$R = \text{diag}(r_1, \dots, r_n), \;\; r_i = 0 \text{ or } 1$$ [3]

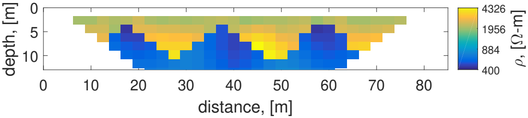
R



Inverted resistivity with RL_1

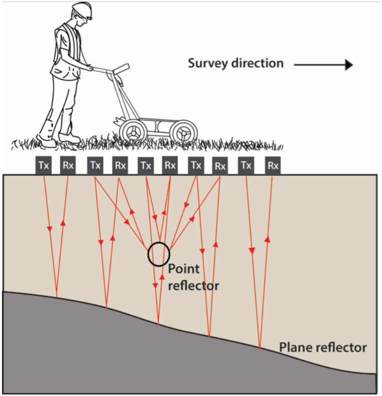


Inverted resistivity with RL_2



[3] Hetrick and M., 2018

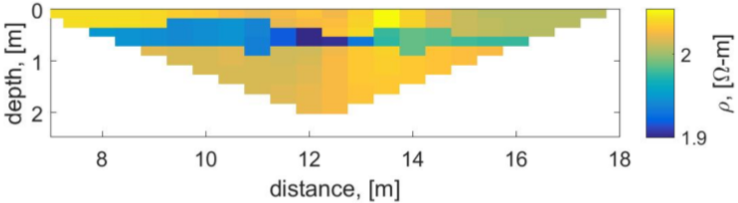
Constraint - Ground Penetrating Radar (GPR)



- GPR survey at BHRS acquired across center of gridded ER survey.
- GPR sampled line collinear with ER survey.
- GPR derived boundary gives constraint for inverting the ER dataset.

Boise Hydrogeophysical Research Site Results

- ER data inverted for resistivity
- Regularization in the form of subsurface boundary constraint inferred from GPR data



Summary - Regularization with Structural Constraints

- Additional data can be used to inform the regularization operator
  - Including initial parameter estimates, or estimates of their first or second derivatives, can always lead to a well-posed problem.
  - Additional derivative information requires less knowledge than initial parameter estimates.
  - Relies on secondary data processing or practitioner interpretation of data.

Assessing Effectiveness of Constraints - Singular Value Expansion

$$\min_x \|d - Ax\|_2^2$$

with solution

$$x = A^\dagger b = \sum_{k=1}^{\infty} \frac{\langle \psi_k, b \rangle}{\sigma_k} \phi_k$$

ψ_k, ϕ_k orthonormal singular functions

$\sigma_k \to 0$ as $k \to \infty$

- Conditioning measured by decay rate of singular values
  - e.g. decay rate $q \Rightarrow \sigma_k$ decays like k^{-q}

Singular Values of Tikhonov Operator

$$\min_x \|d - Ax\|_2^2 + \alpha^2 \|x\|_2^2$$

with solution

$$x = A_\alpha^\dagger b = \sum_{k=1}^{\infty} \frac{\sigma_k}{\sigma_k^2 + \alpha^2} \langle \psi_k, b \rangle \phi_k$$

so that

$$\frac{\sigma_k}{\sigma_k^2 + \alpha^2} \to 0, \text{ as } k \to \infty$$

and α restricts the solution space[4].

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[4] Gockenbach, 2015

Additional Data as Constraints

$$\|d_1 - Ax\|_2^2 + \|d_2 - Bx\|_2^2$$

Singular values satisfy

$$A^*A\phi + B^*B\phi = \sigma^2\phi \quad \text{or} \quad C^*C\phi = \sigma^2\phi$$

and can be approximated with a Galerkin method e.g. $A^{(n)}$, $a_{ij}^{(n)} = \langle q_i, Ap_j\rangle$, where $\{q_i(s)\}_{i=1}^n$ and $\{p_j(t)\}_{j=1}^n$ are orthonormal bases so that

$$A^{(n)} = U^{(n)}\Sigma^{(n)}\left(V^{(n)}\right)^T, \quad \Sigma^{(n)} = \text{diag}\left(\sigma_1^{(n)}, \sigma_2^{(n)}, \ldots \sigma_n^{(n)}\right)$$

Define

$$C^{(n)} = \begin{bmatrix} A^{(n)} \\ B^{(n)} \end{bmatrix}$$

Special case: Self-Adjoint Operators

Use singular functions in Galerkin method

$$a_{ij}^{(n)} = \langle \phi_j, A\phi_i \rangle = \langle \phi_j, \sigma_i \phi_i \rangle = \begin{cases} \sigma_i & i = j \\ 0 & i \neq j \end{cases}$$

then

$$A^{(n)} = \Sigma_A^{(n)} \quad \text{and} \quad B^{(n)} = \Sigma_B^{(n)}$$

so that

$$\left(C^{(n)}\right)^T \left(C^{(n)}\right) = \left(\Sigma_A^{(n)}\right)^2 + \left(\Sigma_B^{(n)}\right)^2$$

and

$$\sigma_i \left(C^{(n)}\right) = \sqrt{\sigma_i \left(A^{(n)}\right)^2 + \sigma_i \left(B^{(n)}\right)^2}$$

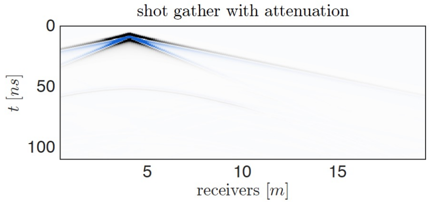
Summary - Combining data

- Simultaneously inverting multiple data sets will necessarily reduce the amount of regularization necessary to resolve ill-posedness.
  - However, adding data may not improve decay rate in individual inversions.
- Singular values from theoretical models indicate properties and or situations where different data types most effectively regularize each other.

Complementary data in Subsurface Imaging

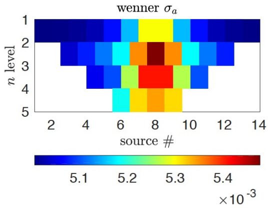
Ground Penetrating Radar

- High frequency
- Conductivity through attenuation and reflection

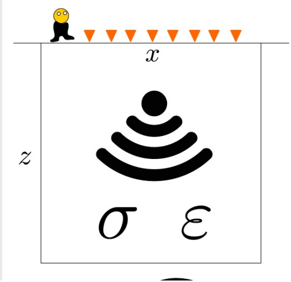


Electrical Resistivity

- Low frequency
- Directly sensitive to conductivity



GPR Model



$$\varepsilon\ddot{u} + \sigma\dot{u} = \frac{1}{\mu}\nabla^2 u + s_w$$ [5]

u-electric field, ε-permittvity

μ-constant permeability

s_w-source wavelet

Model parameters: conductivity σ and permitivity ϵ

Observed data: electric current Mu

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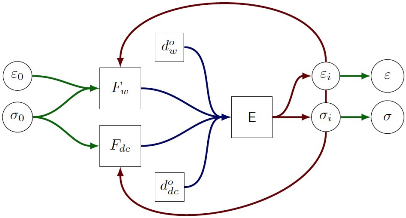
[5] Yee, 1966; Berenger, 1994, Ernst et al., 2007

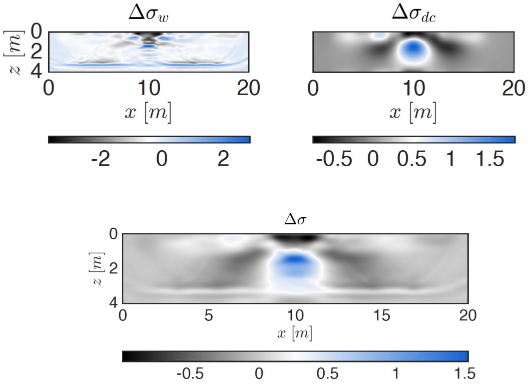
Inverting ER and GPR jointly - full physics



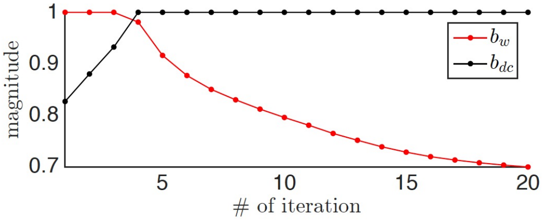
$$\boxed{E = E_w + E_{dc}}$$

$$\varepsilon \leftarrow \varepsilon + \Delta\varepsilon$$
$$\sigma \leftarrow \sigma + \alpha\left(b_w \Delta\sigma_w + b_{dc}\Delta\sigma_{dc}\right)$$

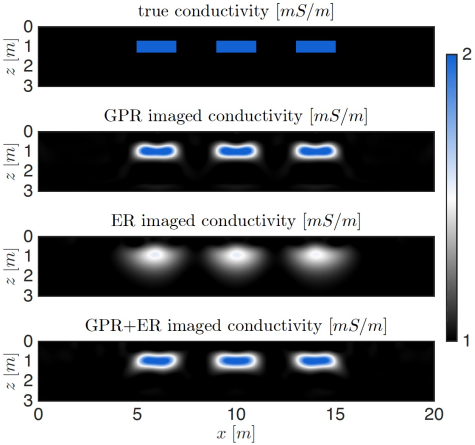
Combining updates



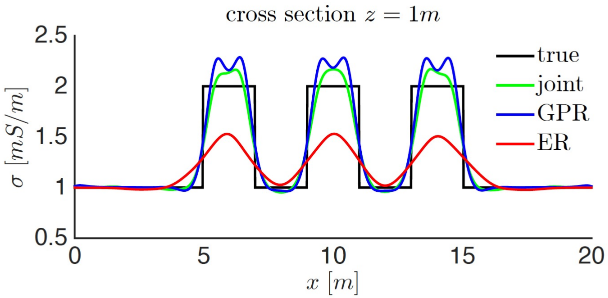
Data weights



Inverted images - full physics



Inverted cross section - full physics



Summary - Joint inversion

- We have developed a joint inversion algorithm to solve for both permittivity ϵ and conductivity σ using complementary GPR and ER data.
  - Full physics that describe the data were incorporated into the inversion.
- Data weights capture the sensitivities of the different physics during the inversion.
- Features were recovered that neither GPR or ER can individually resolve.

Future Work

- Use singular values to quantify the value of combining different data types
  - Compare decay rate of singular values of individual operators to those of joint operators.
- Identify singular values of
  - Green's function solutions of wave and diffusion equations.
  - Covariance matrices that can be used to weight Tikhonov regularization operator.
  - Cross-gradient operator used to identify parameters with similar structure.

Thank you!

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