Data Assimilation and Climate
AIMS Rwanda, March, 2020

Quiz 3

Kalman Filter

Consider the process and data models

$$\begin{aligned}\mathbf{X}(i) &= \mathbf{M}_i\mathbf{X}(i-1) + \boldsymbol{\delta}_i \quad i = 1, \ldots, N \\ \mathbf{Y}(i) &= \mathbf{H}_i\mathbf{X}(i) + \boldsymbol{\epsilon}_i, \quad i = 1, \ldots, N\end{aligned}$$

with $\mathbf{X}(0) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_i)$, $\boldsymbol{\delta}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_i)$. Use the following formulas for the Kalman Filter to answer the questions

$$\begin{aligned}\boldsymbol{\mu}_{i|i-1} &= \mathbf{M}_i\boldsymbol{\mu}_{i-1|i-1} \\ \boldsymbol{\Sigma}_{i|i-1} &= \mathbf{Q}_i + \mathbf{M}_i\boldsymbol{\Sigma}_{i-1|i-1}\mathbf{M}_i^T \\ \mathbf{K}_i &= \boldsymbol{\Sigma}_{i|i-1}\mathbf{H}_i^T\left(\mathbf{H}_i^T\boldsymbol{\Sigma}_{i|i-1}\mathbf{H}_i + \mathbf{R}_i\right)^{-1} \\ \boldsymbol{\mu}_{i|i} &= \boldsymbol{\mu}_{i|i-1} + \mathbf{K}_i\left(\mathbf{y}(i) - \mathbf{H}_i\boldsymbol{\mu}_{i|i-1}\right) \\ \boldsymbol{\Sigma}_{i|i} &= \left(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i\right)\boldsymbol{\Sigma}_{i|i-1}\end{aligned}$$

for $i = 1, \ldots, N$.

1. Let

$$\begin{aligned}x(i) &= x(i-1) + \delta_i \quad i = 1, \ldots, N \\ y(i) &= x(i) + \epsilon_i, \quad i = 1, \ldots, N\end{aligned}$$

with $x(0) \sim \mathcal{N}(\mu, \sigma^2)$, $\epsilon_i \sim \mathcal{N}(0, 0.25)$, $\delta_i \sim \mathcal{N}(0, 1)$. Show that the Kalman gain matrix is

$$K_i = \frac{1 + \Sigma_{i-1|i-1}}{1.25 + \Sigma_{i-1|i-1}}$$

2. Let

$$\begin{aligned}\mathbf{X}(i) &= \mathbf{M}\mathbf{X}(i-1) + \boldsymbol{\delta}_i \quad i = 1, \ldots, N \\ \mathbf{Y}(i) &= \mathbf{H}\mathbf{X}(i) + \boldsymbol{\epsilon}_i, \quad i = 1, \ldots, N\end{aligned}$$

with $\mathbf{X}(0) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, $\boldsymbol{\delta}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$. Assume that the forecasting covariate matrices $\boldsymbol{\Sigma}_{i|i-1}$ converge to a limit $\boldsymbol{\Sigma}_0$. That is, assume there is an I for which $\boldsymbol{\Sigma}_{i|i-1} = \boldsymbol{\Sigma}_0$ for all $i > I$. Derive the equation

$$\boldsymbol{\Sigma}_0 = \mathbf{Q} + \mathbf{M}(\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_0\mathbf{H}^T(\mathbf{H}\boldsymbol{\Sigma}_0\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}\boldsymbol{\Sigma}_0)\mathbf{M}^T$$