Data Assimilation and Climate
AIMS Rwanda, March, 2020

Make-up quiz
3DVar

Consider the cost function

$$\mathcal{J}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}^b) + \frac{1}{2}(\mathbf{H}\mathbf{x}-\mathbf{y})^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{x}-\mathbf{y}),$$

where $\mathbf{x}^b$ is the background state and $\mathbf{y}$ contains the observations, while

$$\begin{aligned} \mathbf{x} &= \mathbf{x}^b + \boldsymbol{\eta} \\ \mathbf{y} &= \mathbf{H}\mathbf{x} + \boldsymbol{\epsilon} \end{aligned}$$

with $\boldsymbol{\eta} \sim \mathcal{N}(0, \mathbf{B})$ and $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \mathbf{R})$. Assume that $\mathbf{x}$ and $\mathbf{y}$ are random variables, and that $\mathbf{x}^b$ is not a random variable.

1. Let the innovation term be denoted by $\mathbf{h} = \mathbf{H}\mathbf{x}^b - \mathbf{y}$.
   (a) Show that $\mathcal{J}(\mathbf{x}^b) = \frac{1}{2}\mathbf{h}^T\mathbf{R}^{-1}\mathbf{h}$.
   (b) Show that $\mathbb{E}[\mathbf{h}] = \mathbf{0}$
2. Recall that the minimum of the cost function occurs at $\hat{\mathbf{x}} = \mathbf{x}^b + \mathbf{K}\left(\mathbf{y} - \mathbf{H}\mathbf{x}^b\right)$, where $\mathbf{K} = \mathbf{B}\mathbf{H}^T\left(\mathbf{H}\mathbf{B}\mathbf{H}^T + \mathbf{R}\right)^{-1}$. Show that $\mathbb{E}[\hat{\mathbf{x}}] = \mathbf{x}^b$.