Data Assimilation and Climate
AIMS Rwanda, March, 2020

Lecture Activities
Introduction to Calculus of Variation

Consider the process model

$$x_{i+1} = \alpha x_i + \delta_i,\ i = 1, 2, 3,$$

data model

$$y_i = x_i + \epsilon_i,\ i = 2, 3.$$

and cost function

$$\mathcal{J}(x_2, x_3; y_2, y_3) = (x_2 - y_2)^2 + (x_3 - y_3)^2.$$

1. Minimize the cost function with respect to x_2. Assume $\delta_i = 0$, so that $x_{i+1} = \alpha x_i$, and find the value x_2^* for which the cost function $\mathcal{J}(x_2; y_2, y_3)$ is minimum. Your formula for x_2^* should not include x_3.

   Solution: $x_2^* = \frac{y_2+\alpha y_3}{1+\alpha^2}$

2. Given the value of x_2^* from 1. use the process model to find the corresponding values of x_3^*, x_4^* and x_1^*. Your formulas should be functions of x_2^*.

   Solution: $x_3^* = \alpha x_2^*$, $x_4^* = \alpha^2 x_2^*$, $x_1^* = \alpha^{-1} x_2^*$.

3. What initial value x_1 should you use in the process model, so that you get the best fit in the data model?

   Solution: $x_1 = \frac{y_2+\alpha y_3}{\alpha(1+\alpha^2)}$