Data Assimilation and Climate
AIMS Rwanda, March, 2020

Lecture Activities
Bayesian approach to Data Assimilation

1. Statistics review.
   Given that

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\, dx \\ \mathrm{Cov}(X,Y) &= E[(X-\mu_x)(Y-\mu_y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f_{XY}(x,y)\, dxdy \end{aligned}$$

   (a) Find $E[c]$ where c is a constant.
   Solution: $E[c] = c$

   (b) Show that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
   Solution: Use the definition above and see that $(x-\mu_x)(y-\mu_y) = (y-\mu_y)(x-\mu_x)$.

   (c) If $\mathbf{Y} = (Y_2, Y_3)^T$, find the general matrix $\mathrm{Cov}(\mathbf{Y})$.
   Solution:

$$\mathrm{Cov}(\mathbf{Y}) = \begin{bmatrix} \mathrm{Var}(Y_2) & \mathrm{Cov}(Y_2,Y_3) \\ \mathrm{Cov}(Y_3,Y_2) & \mathrm{Var}(Y_3) \end{bmatrix}$$

   .

   (d) Suppose $X_1 \sim \mathcal{N}(\mu, \sigma^2)$ and that $X_2 = \alpha X_1$. Then X_2 is also normally distributed. Find the mean and variance of X_2.
   Solution:
   $E[X_2] = E[\alpha X_1] = \alpha E[X_1] = \alpha\mu$,
   $\mathrm{Var}[X_2] = \mathrm{Var}[\alpha X_1] = \alpha^2 \mathrm{Var}[X_1] = \alpha^2\sigma^2$.
   $X_2 \sim \mathcal{N}(\alpha\mu, \alpha^2\sigma^2)$

2. Consider the process model

$$X_{i+1} = \alpha X_i,\ \ i = 1,2,3,$$

   and data model

$$Y_i = X_i + \epsilon_i,\ \ i = 2,3.$$

   along with

$$X_1 \sim \mathcal{N}(\mu_0, \sigma^2), \quad \epsilon_i \sim \mathcal{N}(0, \tau^2).$$

   (a) Find the means of X_2, X_3 and X_4.
   Solution: $E[X_2] = \alpha\mu_0$, $E[X_3] = \alpha^2\mu_0$, $E[X_4] = \alpha^3\mu_0$. Thus $\mathbf{X} = (X_1, X_2, X_3, X_4)$ has mean vector $\boldsymbol{\mu} = (\mu_0, \alpha\mu_0, \alpha^2\mu_0, \alpha^3\mu_0)^T$

   (b) Find the variances of X_2, X_3 and X_4.
   Solution: $\mathrm{Var}(X_2) = \alpha^2\sigma^2$, $\mathrm{Var}(X_3) = \alpha^4\sigma^2$, $\mathrm{Var}(X_4) = \alpha^6\sigma^2$.

   (c) Find $\mathrm{Cov}(X_1,X_2)$, $\mathrm{Cov}(X_1,X_3)$, $\mathrm{Cov}(X_1,X_4)$, $\mathrm{Cov}(X_2,X_3)$, $\mathrm{Cov}(X_2,X_4)$, $\mathrm{Cov}(X_3,X_4)$.
   Solution: $\mathrm{Cov}(X_1,X_2) = \alpha\sigma^2$, $\mathrm{Cov}(X_1,X_3) = \alpha^2\sigma^2$, $\mathrm{Cov}(X_1,X_4) = \alpha^3\sigma^2$, $\mathrm{Cov}(X_2,X_3) = \alpha^3\sigma^2$, $\mathrm{Cov}(X_2,X_4) = \alpha^4\sigma^2$, $\mathrm{Cov}(X_3,X_4) = \alpha^5\sigma^2$.

(d) Use your answers to (**??**) and (**??**) to form the covariance matrix $\mathbf{\Sigma}_X$.
Solution:

$$\mathbf{\Sigma}_X = \sigma^2 \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \\ \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \end{bmatrix}$$

3. Assume we have process and data models for which

- the *prior distribution* is $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \sigma_x^2 \mathbf{I})$, $\mathbf{X} \in \mathbb{R}^2$
- the conditional distribution is $\mathbf{Y}|\mathbf{X} \sim \mathcal{N}(\mathbf{X}, \sigma_y^2 \mathbf{I})$, $\mathbf{Y} \in \mathbb{R}^2$

and we use the the Bayesian method to assimilate data.

(a) Find the gain matrix $\mathbf{K}$.
Solution: $\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}\mathbf{I}$

(b) Find the mean $\boldsymbol{\mu}^*$ of the posterior distribution $\mathbf{X}|\mathbf{Y}$, and discuss how the data informed the state estimates.
Solution:

$$\boldsymbol{\mu}^* = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

State estimates are the data scaled by $\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}$.

(c) Find covariance matrix $\mathbf{\Sigma}^*$ of the posterior distribution $\mathbf{X}|\mathbf{Y}$ and discuss the meaning its elements.
Solution:

$$\mathbf{\Sigma}^* = \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}\mathbf{I}$$

The first diagonal element of $\mathbf{\Sigma}^*$ is $\text{Var}(X_1|\mathbf{Y})$ and the second diagonal element is $\text{Var}(X_2|\mathbf{Y})$. The variance of both state estimates are the same. Since the off diagonal elements are zero, the estimates $x_1^* = \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} y_1$ and $x_2^* = \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} y_2$ have zero covariance.